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The stability of general relativistic cosmological theory

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Abstract. We analyse the behaviour of homogeneous and isotropic solutions to a gravity theory that arises from the variation of an arbitrary analytic function of the space–time scalar curvature. Such a theory generalises Einstein's general relativity wherein this function is linear in the curvature. We prove conditions for the existence and stability of the general relativistic de Sitter and Friedman solutions within the general theory, prove necessary and sufficient conditions for the existence of cosmological singularities and particle horizons and analyse the asymptotic behaviour of ever-expanding universe models. The conditions under which Minkowski space–time and Schwarzschild space–time are stable is investigated and their instability, together with the pathological behaviour of certain cosmological models, traced back to the non-minimality of the stationary action giving rise to the field equations. The significance of these results for quantum theories of gravity and the 'inflationary' model of the early universe is discussed.

1. Introduction

There have been many investigations into the stability of particular solutions of Einstein's general relativity theory (GR) within the space of all solutions of the theory. Such studies are of particular interest in deciding how galaxies formed (Lifshitz 1946, Barrow 1980, Bardeen 1980, Peebles 1981) and also generate interesting mathematical problems concerning the relationship between true solutions of GR and approximations that neighbour solutions with special symmetries (Barrow and Tipler 1979). In this paper we shall examine the existence and stability of homogeneous and isotropic cosmological solutions to GR with respect, not to perturbations within GR, but to perturbations of GR into a large class of metric gravity theories. We shall show that the familiar de Sitter and Friedman solutions almost always exist and, under fairly general circumstances, are stable.

Einstein's field equations of GR were first derived from an action principle by Hilbert in 1915, but this property is not peculiar to Einstein's theory. Indeed Poisson's equation for the Newtonian gravitational potential $\phi(\boldsymbol{x}, t)$ can be obtained by varying a Newtonian action functional S_N with respect to ϕ, if we choose

$$S_N = -\int [\rho v^2 - \rho\phi - (\nabla\phi)^2]\, d^3x\, dt \qquad (1.1)$$

where v is the material velocity field, and ρ the mass density ($8\pi G \equiv 1$ and $c = 1$ for the velocity of light throughout).

Einstein's Lagrangian for GR is a linear function of the four-curvature of space–time R:

$$L_E = -2\Lambda + R \tag{1.2}$$

where Λ is a constant to be identified with the 'cosmological constant', the Einstein action is (see § 2 for notational details)

$$S_E = -\tfrac{1}{2}\int d^4x\,\sqrt{-g}L_E. \tag{1.3}$$

Since Hilbert's recognition of this fact, there have been many attempts to create a natural generalisation of GR by considering the field equations that result upon varying an action functional that contains curvature invariants of higher than linear order in (1.2) (Lanczos 1938, Weyl 1921, Eddington 1924, Buchdahl 1948, Pais and Uhlenbeck 1950, Utiyama and De Witt 1962, Havas 1977, Stelle 1978). These particular investigations all considered the effect of '*quadratic Lagrangians*' involving some of the four possible *second-order* curvature invariants that can be created from the scalar, Ricci and Riemann curvature tensors: these are R^2, $R_{ab}R^{ab}$, $R_{abcd}R^{abcd}$ and $\varepsilon^{iklm}R_{iksl}R^{st}_{lm}$ where ε^{iklm} is the completely antisymmetric tensor of rank four. These investigations, and those that arise from them (Ruzmaikin and Ruzmaikina 1970, Giesswein *et al* 1974, Nariai and Tomita 1971, Gurovich 1977), had a two-fold purpose: first, as there is no *a priori* reason to restrict the gravitational Lagrangian to a linear function of R, one might expect that by including higher powers of R and related invariants, a more realistic representation of gravitational fields near curvature singularities (where $R \to \infty$) would be obtained. Perhaps singularities would disappear from these generalised theories? Second, some quantum corrections to Einstein's GR are equivalent to augmenting the Einstein Lagrangian by higher-order curvature invariants (Sakharov 1967, Fischetti *et al* 1979, Horowitz and Wald 1978, Stelle 1978, Weinberg 1979). This might lead us to expect that higher-order Lagrangians, subject to suitable constraints, would create a first approximation to some quantised theory of gravity. Analogously, we know that the quantum electrodynamical phenomenon of vacuum polarisation was described by introducing into the Lagrangian nonlinear combinations of the terms originally appearing linearly in the classical theory; in this case the classical electromagnetic field ($\boldsymbol{E}, \boldsymbol{H}$) Lagrangian L (Pechlaner and Sexl 1966)

$$L = \tfrac{1}{2}(E^2 - H^2) \tag{1.4}$$

generalises, with first-order quantum corrections, to include nonlinear combinations of the terms appearing in L, and becomes

$$L' = \tfrac{1}{2}(E^2 - H^2) + (2\alpha_e^2/45m_e^4)[(E^2 - H^2)^2 + 7(\boldsymbol{E}\cdot\boldsymbol{H})^2] \tag{1.5}$$

(where α_e is the fine structure constant and m_e the electron mass).

Most analyses of the gravity theories that arise from nonlinear Lagrangians have confined attention to Lagrangians that contain, at most, quadratic curvature invariants. However, the qualitative differences between the predictions of the quadratic theory compared with GR would lead one to expect further, equally large, changes if cubic curvature invariants were added to the quadratic theory and so on. On approach to a space–time singularity curvature invariants of all polynomial orders ought to play an important dynamical role and will, collectively, decide whether or not a real curvature singularity occurs.

In this paper we shall examine the cosmological models that result from a gravitational Lagrangian that is an arbitrary function of the scalar curvature, $f(R)$. One of the advantages of this choice, aside from its obvious generality compared with earlier investigations, is that it also allows non-polynomial Lagrangians to be considered. Although polynomial Lagrangians might be anticipated to provide a reasonable approximation to the complete Lagrangian when the curvature is small (just as Einstein's choice L_E in (1.2) does), when it becomes large, quantum corrections to L_E could lead to an effective Lagrangian whose predictions differ significantly from those of any theory with a polynomial Lagrangian. Our choice of $f(R)$ is not completely general, of course, because we exclude contributions from any curvature invariants other than R. To include them would be prohibitively complicated because the number of curvature invariants of dimension exceeding $(\text{length})^{-2n}$ increases very rapidly with n. The only limitation we shall need to assume in order to derive many of our results is that f be analytic; that is, it must possess a Taylor series expansion about any point. These restrictions are clearly not totally satisfactory, but they do enable us to take some first steps towards considering the effects of, as yet, uninvestigated higher-order curvature contributions.

There are two wider applications of our results that are worth mentioning at this stage: our investigation into the existence and stability of the de Sitter vacuum solution of GR against terms introduced by the higher-order theories gives us information about the likely stability of 'inflationary' phases during the very early Universe (Guth 1981, Linde 1982, Albrecht and Steinhardt 1982, Barrow and Turner 1982) in the presence of quantum gravitational corrections. Additionally, we note that there has recently been some interest (Nielsen 1981, Barrow 1983b) in the idea that there are really no laws of physics at all—that the Lagrangians of physical interactions are *stochastic* functions with the property that local gauge invariances (equivalent to conservation laws) are well approximated in the low-energy limit. Our investigation of arbitrary gravitational Lagrangians could be viewed in this light. In essence, our investigations are asking the question—suppose a completely arbitrary Lagrangian function of the curvature is chosen as the basis of a gravitation theory; how likely is it that the resulting theory has the Friedman solution, or something very much like it, as a stable solution at late times far from the initial singularity (if indeed there was initially such a singularity)?

In § 2 we introduce the necessary definitions, notation and formalism together with some useful general equations. In § 3 we shall prove an existence theorem for the de Sitter cosmological solution of GR to exist as a solution of the general $f(R)$ Lagrangian theory of gravity and then examine the stability properties of any de Sitter solution that does exist. Section 4 examines the existence and stability of the Friedman solutions of GR in $f(R)$ theories. We also investigate the asymptotic behaviour of these Friedman solutions for both large and small cosmic times and derive the conditions on f necessary for an initial big bang singularity. In § 5, the existence and stability of Einstein's static universe and flat Minkowski space are examined and the conditions for the stability of Minkowki space related to the behaviour of cosmological models in $f(R)$ theories. Finally, in § 6, some brief conclusions are drawn.

2. Mathematical formalism

In this section the notation and key equations required for subsequent analyses will be given. Our GR sign conventions follow those of Weinberg (1972), so the metric signature is $(-+++)$; the Riemann tensor is defined by

$$R^a_{\ bcd} = \Gamma^a_{bc,d} - \Gamma^a_{bd,c} + \Gamma^e_{bc}\Gamma^a_{de} - \Gamma^e_{bd}\Gamma^a_{ce} \tag{2.1}$$

and the Ricci tensor by $R_{ab} = R^c_{\ acb}$. Latin indices run over 0, 1, 2, 3 and we shall adopt units such that $8\pi G \equiv c \equiv 1$.

For the reasons outlined in § 1, we shall study the gravity theory that arises from a Lagrangian that is an analytic function of the Ricci scalar alone; so the gravitational action is:

$$S_G = -\tfrac{1}{2}\int \sqrt{-g} f(R)\, \mathrm{d}^4x. \tag{2.2}$$

This class includes GR with non-zero cosmological constant when

$$f(R) = -2\Lambda + R. \tag{2.3}$$

Moreover, any quadratic Lagrangian leading to an isotropic, homogeneous cosmological model is, in general, equivalent to the choice

$$f(R) = -2\Lambda + R - \tfrac{1}{6}\alpha R^2 \tag{2.4}$$

with α an arbitrary real constant. This simplification arises because the remaining quadratic curvature invariants can be expressed in terms of R^2 by using the following two identities, which hold for any four-dimensional space–time (De Witt 1965)

$$(\delta/\delta g_{ab})\int \mathrm{d}^4x\, \sqrt{-g}(R^{abcd}R_{abcd} - 4R^{ab}R_{ab} + R^2) = 0 \tag{2.5}$$

$$(\delta/\delta g_{ab})\int \mathrm{d}^4x\, \sqrt{-g}\,\varepsilon^{abcd}R_{abef}R_{cd}^{\ \ ef} = 0 \tag{2.6}$$

and, in addition, for isotropic and homogeneous space–times, the identity

$$(\delta/\delta g_{ab})\int \mathrm{d}^4x\, \sqrt{-g}(3R^{ab}R_{ab} - R^2) = 0. \tag{2.7}$$

If the space–time contains matter with action functional S_M the energy–momentum tensor of the matter is

$$T_{ab} = (-g)^{-1/2}\delta S_M/\delta g_{ab} \tag{2.8}$$

and the field equations arise on variation of the total action $S_G + S_M$. From (2.2) and (2.8) we have (Buchdahl 1970)

$$0 = T_{ab} + f'R_{ab} - \tfrac{1}{2}fg_{ab} + f''(\nabla_a\nabla_b R - \Box R g_{ab}) + f'''(\nabla_a R\nabla_b R - \nabla^c R\nabla_c R g_{ab}) \tag{2.9}$$

where

$$\Box \equiv g_{ab}\nabla^a\nabla^b \tag{2.10}$$

and ∇_a is the covariant differential operator.

The field equations (2.9) are obtained by varying the metric alone ('g variation'), with the connection always defined by the Christoffel relation. The Palantini variation, in which the metric and connection are varied independently, is plagued with difficulties when the Lagrangian is nonlinear in R (Buchdahl 1979). Notice that the form of (2.9) is influenced only by the first three derivatives of $f(R)$.

Our discussions will be almost entirely concerned with homogeneous and isotropic cosmological solutions to the field equations (2.9). The standard Friedman metric of GR is

$$ds^2 = -dt^2 + a^2(t)[dr^2/(1-\sigma r^2) + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2] \tag{2.11}$$

where the spatial three-sections are closed, flat or open according as $\sigma = +1, 0, -1$. For this metric we find ($\alpha, \beta = 1, 2, 3$)

$$R_{00} = 3\ddot{a}a^{-1} \qquad R_{\alpha\beta} = -[(a\ddot{a} + 2\dot{a}^2 + 2\sigma)a^{-2}]g_{\alpha\beta} \tag{2.12}$$

$$R = -6a^{-2}(a\ddot{a} + \dot{a}^2 + \sigma). \tag{2.13}$$

If we note the useful relations

$$\Box R = -(\ddot{R} + 3\dot{a}a^{-1}\dot{R}) \tag{2.14}$$

$$\nabla_c R \nabla^c R = -\dot{R}^2 \tag{2.15}$$

then the field equations (2.9) can be found explicitly. Symmetry dictates that only two of these equations are independent, which we take to be the trace of (2.9) and the (00) component. These are (Kerner 1982)

$$Rf' - 2f + 3f''(\ddot{R} + 3\dot{a}a^{-1}\dot{R}) + 3f'''\dot{R}^2 + T = 0 \tag{2.16}$$

$$f'R_{00} + \tfrac{1}{2}f - 3f''\dot{a}a^{-1}\dot{R} + T_{00} = 0. \tag{2.17}$$

It can, furthermore, be verified that (2.16) follows from the differentiation of (2.17) with respect to t, so (2.17) is the only necessary field equation. In terms of the cosmological scale factor $a(t)$ it reads

$$f''(R)[a^2\dot{a}\dddot{a} + a\dot{a}^2\ddot{a} - 2\dot{a}^4 - 2\dot{a}^2\sigma] + \tfrac{1}{6}f'(R)a^3\ddot{a} + \tfrac{1}{36}f(R)a^4 + \tfrac{1}{18}a^4 T_{00} = 0. \tag{2.18}$$

We shall confine our attention to cosmologies containing matter with a perfect fluid equation of state relating the pressure p to the energy density ρ, so

$$p = \gamma\rho = \gamma T_{00} \tag{2.19}$$

and thus

$$T_{00} = \rho_0 a^{-3(\gamma+1)} \qquad \rho_0 \qquad \text{constant.} \tag{2.20}$$

In the particular case of a quadratic Lagrangian and no cosmological constant (equation (2.4) with $\Lambda = 0$), the field equation reduces to the form

$$a^{-2}(\dot{a}^2 + \sigma) + \alpha a^{-4}(2a^2\dot{a}\dddot{a} + 2a\dot{a}^2\ddot{a} - a^2\ddot{a}^2 - 3\dot{a}^4 - 2\dot{a}^2\sigma + \sigma^2) = \tfrac{1}{3}\rho_0 a^{-3(\gamma+1)}. \tag{2.21}$$

The order of this equation can be lowered by eliminating the independent variable (Ruzmaikin and Ruzmaikina 1970). Introducing $b(x)$ where

$$b = (\dot{a}a)^{3/2} \tag{2.22}$$

$$x = a^3(2\sqrt{3})^{-3/2} \tag{2.23}$$

and denoting db/dx by b', (2.21) reduces to

$$b'' - (1/\sqrt{3})b^{-1/3}x^{-4/3}\sigma + b^{-5/3}x^{-2/3}\sigma^2$$
$$+ \alpha^{-1}(b^{-1/3}x^{-2/3} + 2\sqrt{3}b^{-5/3}\sigma - \tilde{\rho}_0 b^{-5/3}x^{-\gamma-1/3}) = 0 \tag{2.24}$$

where

$$\tilde{\rho}_0 \equiv 2^{(1-3\gamma)/2}3^{(1-3\gamma)/4}\rho_0. \tag{2.25}$$

In subsequent sections we shall be interested in the existence and stability of solutions to equations (2.18), (2.21) and (2.24). To conclude our preliminary derivations we write down the general stability equation for solutions of (2.18). We suppose that $a_0(t)$ is a particular exact solution of (2.18) and look for solutions of the form

$$a(t) = a_0(t)[1+\varepsilon(t)] \qquad |\varepsilon(t)| \ll 1. \tag{2.26}$$

Linearising (2.18) about the exact solution, we obtain an ordinary differential equation for $\varepsilon(t)$:

$$A\dddot{\varepsilon} + B\ddot{\varepsilon} + C\dot{\varepsilon} + D\varepsilon = 0 \tag{2.27}$$

where

$$A \equiv -6f_0''a_0^3\dot{a}_0 \qquad B \equiv 6a_0^2(a_0\ddot{a}_0f_0'' - a_0\dot{a}_0\dot{R}_0f_0''' - 4f_0''\dot{a}_0^2) \tag{2.28a, b}$$

$$C \equiv a_0(2a_0^2\dot{a}_0f_0' + a_0^3\dot{R}_0f_0'' + 12\sigma\dot{a}_0f_0'' + 24\dot{a}_0^3f_0'' - 24a_0\dot{a}_0^2\dot{R}_0f_0''') \tag{2.28c}$$

$$D \equiv -2\sigma a_0^2f_0' - 12\sigma\ddot{a}_0a_0f_0'' - 24\sigma\dot{a}_0^2f_0'' + 12\sigma a_0\dot{a}_0\dot{R}_0f_0''' + (\gamma+1)\rho_0a_0^{1-3\gamma} \tag{2.28d}$$

where R_0 is defined as a solution of (2.13) with $a \equiv a_0$; $f_0 \equiv f(R_0)$, $f_0' \equiv f'(R_0)$ and so on, and we have assumed that $f(R)$ can be expanded in a Taylor series in $(R - R_0)$.

3. de Sitter universes

de Sitter's solution to the vacuum Einstein equations with cosmological constant has proven to be of fundamental significance and recurrent physical interest; the first expanding universe model, it later provided a dynamical description of the steady-state cosmology and, most recently, it has emerged as a model for an 'inflationary' phase of the very early Universe. This last role has been created by the unusual phase portraits of grand unified gauge theories at high energy. As discussed by many authors (Guth 1981, Sato 1981, Linde 1982, Hawking and Moss 1982, Albrecht and Steinhardt 1982, Barrow and Turner 1982), if the phase transition associated with the spontaneous breakdown of symmetry in a grand unified theory is first order, then a portion of the universe may find itself residing in a metastable symmetric vacuum state after the material cools below the grand unification energy, m_u. This symmetric vacuum has an energy of order m_u^4. As the Higgs field evolves slowly from this metastable vacuum state to the true, but asymmetric vacuum state of lower energy, the vacuum energy released dominates the cosmological dynamics and forces the scale factor $a(t)$ to grow exponentially with proper time:

$$a(t) = \exp(t/t_u) \qquad t_u \simeq m_p m_u^{-2} \tag{3.1}$$

where $m_p \simeq 10^{19}$ GeV is the Planck mass. If the Universe remains in this de Sitter phase for long enough ($\gtrsim 70t_u$), then an explanation can be offered for the proximity of the present expansion to the Euclidean, Einstein–de Sitter form, its spatial homogeneity over dimensions larger than a gigaparsec, and the absence of an overwhelming cosmic abundance of magnetic monopoles with mass close to m_u.

When the vacuum energy difference dominates the dynamics it has a Lorentz invariant stress tensor equivalent to that of a perfect fluid with equation of state

$$p_v = -\rho_v. \tag{3.2}$$

The stress tensor is

$$(T_{ab})_v = p_v g_{ab}. \tag{3.3}$$

The continuity equation for the evolution of such a fluid shows p_v is constant, and so (3.3) is equivalent to adding a cosmological constant term to the Einstein equations, of the form Λg_{ab}.

During the period when the Λ term dominates the cosmic expansion it can be shown that no scalar, vector or tensor perturbations to the de Sitter evolution grow in time (Barrow 1983a, Gibbons and Boucher 1983); that is, the de Sitter solution is stable although not, in fact, asymptotically stable.

Starobinskii (1980) has proposed a model in which m_u equals m_p and inflation commences at the Planck epoch. He, and others, have imagined a de Sitter state evolving from $t = -\infty$ until a phase transition occurs, whereupon it transforms into a Friedman universe. To examine the consistency of such a picture we must investigate the stability of de Sitter space to quantum gravitational corrections. With this in mind, we shall prove criteria for the existence, uniqueness and stability of de Sitter universes in gravity theories with an $f(R)$ Lagrangian.

3.1. Existence

We require conditions for the existence of a maximally symmetric vacuum solution of the gravitational field equations; therefore, the Riemann tensor can be written

$$R_{ijkl} = \tfrac{1}{12}R\,(g_{ik}g_{jl} - g_{il}g_{jk}) \tag{3.4}$$

$$R_{ik} = \tfrac{1}{4}R g_{ik}. \tag{3.5}$$

From the Bianchi identities it follows that R is covariantly constant, so we have, say,

$$R \equiv R_0. \tag{3.6}$$

Using (2.9), these constraints (3.4)–(3.6) yield a simple existence condition:

$$R_0 f'(R_0) = 2f(R_0). \tag{3.7}$$

Thus, given any $f(R)$ gravity theory, if there exists a solution R_0 of (3.7) then the theory contains the GR de Sitter solution with constant curvature R_0 and with metric scale factor $a(t)$ identical to the form taken in GR. In the Einstein case, (1.2), we have simply

$$R_0 = 4\Lambda. \tag{3.8}$$

The result (3.7) has a number of interesting consequences. The solution need not be unique, for example $f(R)$ equal to $\sin R$ leads to solutions with R_0, given by the countable infinity of solutions to

$$\tan R_0 = R_0. \tag{3.9}$$

Equation (3.7) is identically satisfied by the purely quadratic Lagrangian theory; that is, solutions exist for any R_0. For the pure power-law Lagrangian

$$f(R) = AR^n \qquad n > 2 \tag{3.10}$$

there is only a solution for R_0 equal to zero. Finally, we remark that choices exist for $f(R)$ for which there is no solution of (3.7), for example $\exp(-R^2)$.

As an aside, we can, in the spirit of the stochastic gauge theorists mentioned in § 1, ask how 'probable' it is that a randomly chosen $f(R)$ gravity theory possess a de Sitter solution. When this question is suitably framed it can be answered quantitatively. Suppose we limit ourselves to polynomial Lagrangians of the form

$$f(R)=\sum_{n=0}^{N} a_n R^n; \tag{3.11}$$

then solving (3.7) reduces to finding real solutions of the polynomial equation

$$\sum_{n=0}^{N} (2-n)a_n R_0^n = 0. \tag{3.12}$$

Kac (1959) has considered the problem of calculating the probability that an nth-order polynomial with normally distributed coefficients possesses a real root. If the a_i in (3.11) and (3.12) are normally distributed, $N(0,1)$, then the relative probability that (3.12) possesses n real roots (each will correspond to a constant curvature solution of the field equations) is

$$\begin{aligned} P_n &= \left\{4\pi^{-1}\int_0^1\left[\left(\sum_0^{n-1} t^{2k}\right)\left(\sum_1^{n-1} k^2 t^{2k-2}\right)-\left(\sum_1^{n-1} kt^{2k}\right)^2\right]\mathrm{d}t\right\}\Big/\sum_0^{n-1} t^{2k} \\ &\sim 2\pi^{-1}\log n \qquad \text{as } n\to\infty. \end{aligned} \tag{3.13}$$

This quantifies the simple fact that it is relatively rare for a polynomial of high order to possess real rather than complex pairs of roots. The distribution of their values peaks at +1 and −1, reflecting the fact that (3.12) will be satisfied when each term is, on average, of similar magnitude but opposite sign.

3.2. Stability

We have shown that a general $f(R)$ theory will often contain the GR de Sitter solution as a particular case. Now we wish to examine the conditions under which an extant solution of this type will be stable.

It is sufficient to examine the zero curvature ($\sigma=0$) model†. In (2.26)–(2.28) we perturb the expansion scale factor so

$$a(t)=\mathrm{e}^{mt}(1+\varepsilon(t)) \qquad |\varepsilon|<1 \tag{3.14}$$

and we have, therefore,

$$R_0 \equiv -12m^2 \tag{3.15}$$

and

$$f_0''\dddot{\varepsilon}+3mf_0''\ddot{\varepsilon}-\tfrac{1}{3}\dot{\varepsilon}(f_0'+12m^2 f_0'')=0. \tag{3.16}$$

Solutions to (3.16) of the form $\exp(\lambda t)$ exist for

$$\lambda=0 \qquad -\tfrac{3}{2}m\pm\tfrac{1}{2}[25m^2+(4f_0'/3f_0'')]^{1/2}. \tag{3.17}$$

Thus, there can exist both growing and decaying perturbations to the GR de Sitter solution in general. An instructive example is the quadratic Lagrangian theory (2.4), where (3.17) shows that de Sitter solutions will be stable if $\alpha<0$, but unstable if $\alpha>0$.

† The $\sigma=\pm1$ models correspond to different time slicings through the $\sigma=0$ model.

We shall now proceed to examine this stability in more detail. In the reduced variables (2.22)–(2.23) the general ($\sigma \neq 0$) de Sitter solution to GR has the form

$$b(x) = x^{1/2}(\tilde{\rho}_0 x^{2/3} - 2\sqrt{3}\sigma)^{3/4} \tag{3.18}$$

$$\tilde{\rho}_0 = -12\Lambda. \tag{3.19}$$

Again, it is sufficient to examine the stability of the $\sigma = 0$ model in detail. Setting

$$b(x) = \tilde{\rho}_0^{3/4} x f(x) \tag{3.20}$$

and changing variables to

$$x = e^z \tag{3.21}$$

(the dot denotes d/dz) we find that $f(x)$ must satisfy

$$\ddot{f} + \dot{f} + K(f^{-1/3} - f^{-5/3}) = 0 \qquad K \equiv -12\Lambda\alpha^{-1}. \tag{3.22}$$

The constant K is an arbitrary parameter measuring the relative importance of the vacuum stress (Λ) and the nonlinear portion of the Lagrangian. If we introduce $u(x)$ by writing (3.22) in the form

$$\dot{f} = u \qquad \dot{u} = -u - K(f^{-1/3} - f^{-5/3}) \tag{3.23}$$

then the critical points of this system occur when (u, f) is (0, 0) or (0, 1). The physically real critical point is (0, 1), and it corresponds to the de Sitter solution (3.18) with $\sigma = 0$. If we put $f = 1 + \eta$ to move this point to the origin and linearise, then (3.23) become

$$\dot{\eta} = u \qquad \dot{u} = -u - \tfrac{4}{3}K\eta. \tag{3.24}$$

The characteristic equation of this system has roots

$$\lambda = \tfrac{1}{2}[-1 \pm (1 - \tfrac{16}{3}K)^{1/2}]. \tag{3.25}$$

This is equivalent to the general form (3.17) for quadratic $f(R)$. We see there arè two distinct cases depending on the sign of K.

$K > 0$. The critical point (0, 1) is a *stable focus* (see figure 1).
As $z \to \infty$ so $x \to \infty$ and $a \to \infty$, and all solutions asymptotically approach the de Sitter solution of GR. If we alter the direction of time we see that the GR de Sitter solution will be unstable as $x \to -\infty$. In this case the attracting focus in the phase portrait bifurcates either into a repelling node, for $K \in (\frac{3}{16}, \infty)$, or a repelling focus for $K \in (0, \frac{3}{16})$.
$K < 0$. The singular point (0, 1) corresponding to the de Sitter solution of GR is a saddle point. The linearised behaviour in its vicinity is readily obtained. The full phase portrait is shown in figure 2.

We see that all but a set of measure zero of the solutions deviate from the GR de Sitter point, (0, 1), both as $t \to +\infty$ and $t \to -\infty$. To illustrate the general behaviour specifically we have found an exact solution of the system (3.22) when $K = -3$ (it is indicated in figure 2)

$$f(z) = (4z + B)^{3/4} \qquad B \text{ constant.} \tag{3.26}$$

It represents a typical late-time trajectory of the solution with $a(t)$ of the form

$$a(t) \propto \exp[(\tilde{\rho}_0^{1/2} t/4\sqrt{3} - \beta)^2] \qquad \beta \text{ constant.} \tag{3.27}$$

This behaviour is clearly pathological as $t \to \infty$, since expansion as $a \propto \exp(t^2)$ for $t \to \infty$

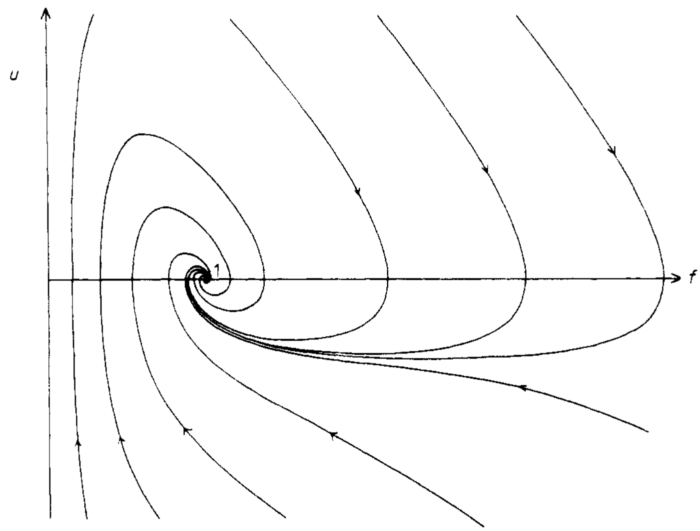

Figure 1. The phase-plane portrait of the system (3.23) when $K > 0$. Arrows denote the direction of increasing cosmic time. The critical point at (0, 1) corresponds to the zero curvature, ($\sigma = 0$), de Sitter space–time.

leads to a scalar curvature singularity $R \sim \ddot{a}a^{-1} \to \infty$ as $t \to \infty$. Therefore, in the cases covered by figure 2, the late-time behaviour is dominated by the influence of the quadratic curvature terms. This is a classic example of a *singular perturbation* problem: we have added higher-order curvature terms to the GR theory which, according to dimensional analysis alone, should become increasingly insignificant as $a \to \infty$, however, quite the opposite is true; see § 6 for further discussion of this point.

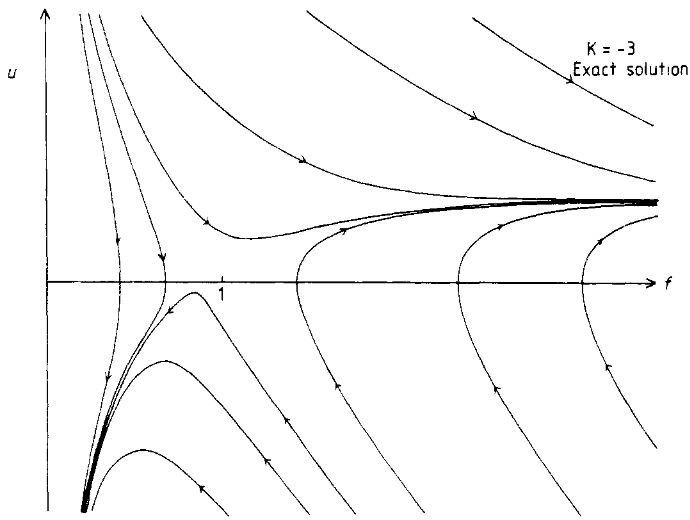

Figure 2. The phase-plane portrait of the system (3.23) when $K < 0$. Arrows denote the direction of increasing cosmic time. The critical point (0, 1) corresponds to the zero curvature, ($\sigma = 0$), de Sitter space–time. The particular exact solution for $K = -3$, given by equation (3.26), is indicated.

3.3. *Asymptotic behaviour*

In order to gain some further perspective on the role played by the nonlinear curvature terms in the Lagrangian as $t \to \infty$ we shall examine the asymptotic behaviour of (2.24) when $\gamma = -1$. This corresponds to the late-time evolution of a vacuum, homogeneous and isotropic universe with a cosmological constant. The neglect of the matter terms is admissible because they become negligible for large t compared with other terms retained in (2.24). In the limit of large α, in which the quadratic Lagrangian totally dominates the GR terms, the asymptotic behaviour is determined by ($b' = \mathrm{d}b/\mathrm{d}x$)

$$b'' - \tfrac{1}{3}\sqrt{3}\sigma b^{-1/3} x^{-4/3} + b^{-5/3} x^{-2/3} \sigma^2 = 0. \tag{3.28}$$

This is identical to the equation governing the behaviour of isotropic and homogeneous cosmological solutions to a quadratic Lagrangian theory with $\Lambda = 0$ and $\gamma < \frac{1}{3}$ in the opposite limit, $t \to 0$ (compare Frenkel and Brecher 1982).

For $\sigma = 0$ the general solution to (3.28) satisfying the necessary boundary condition $b(0) = 0$ imposed by the definitions (2.22)–(2.23) is

$$b(x) = Ax \qquad A \text{ constant.} \tag{3.29}$$

This is simply the de Sitter solution studied in § 3.2.

When $\sigma = \pm 1$, equation (3.28) possesses the special constant solutions

$$b(x) = (4 - 2\sigma)^{3/4} 3^{-3/8} x^{1/2}. \tag{3.30}$$

It is straightforward to show that these constant solutions are always unstable solutions of (3.28).

4. Friedman universes

The Friedman cosmological model is the paradigm of modern theoretical cosmology. It provides an excellent description of the observable universe. However, it makes an unusual prediction regarding the state of the universe in the past; this prediction it shares with many other physically realistic cosmological solutions of GR (Hawking and Ellis 1973): it predicts a space–time singularity in our past. In the Friedman models this singularity is accompanied by infinities in the matter density and scalar curvature. This degeneracy has motivated several studies of cosmological models in quadratic Lagrangian theories with the hope that the inclusion of higher curvature terms would exorcise the curvature singularities and produce space–times that are geodesically complete to the past and future. In 1970 Ruzmaikin and Ruzmaikina discovered some interesting properties of the *spatially flat*, radiation-filled, homogeneous and isotropic solutions to the quadratic Lagrangian theory (2.4) with $\sigma = 0$. They found that it was possible to avoid an initial singularity, but, if they did so, the models failed to approach the Friedman solution of GR ($a \propto t^{1/2}$), as $t \to \infty$. In fact, the situation was far worse: those solutions avoiding a singularity as $t \to 0$ experience a curvature singularity in the future, $R \to \infty$ as $t \to \infty$! We shall see that such pathologies are connected with the instability of flat Minkowski space and this renders such solutions physically inadmissible. If the flat, radiation solutions of the quadratic Lagrangian theory do possess an initial singularity then, at late times, they approach the standard Friedman model of GR. Similar conclusions were drawn for flat models by Nariai and Tomita (1971), and these analyses were duplicated by

Giesswein *et al* (1974). Subsequently, Giesswein and Streeruwitz (1975) also examined the evolution of some radiation models with positive and negative curvature using numerical methods. Fischetti *et al* (1979) have performed analyses of the flat quadratic solutions in the context of quantum cosmology, and they find that it is possible to have solutions which are horizon-free near the singularity. These results have been rediscovered recently by Frenkel and Brecher (1982). In another sequence of papers Gurovich (1977) has examined the influence of logarithmic terms on the quadratic Lagrangian theory. He finds that, by adding a term to the Lagrangian of the form $R^2 \ln(R/R_p)$, where R_p is constant, models can be constructed that are singularity-free as $t \to 0$ and yet which also approach the GR Friedman solution at late times. This is possible because the logarithmic term in the Lagrangian can change its sign as R is greater or less than R_p. As we have seen in § 3.2, it is the *sign* of the quadratic contributions to the curvature that controls the behaviour. Recall that our analyses are confined to analytic $f(R)$ Lagrangians; although most of our results can be extended to include non-analytic cases like Gurovich's we shall not give the details of these extensions.

There have been a small number of investigations into the behaviour of anisotropic cosmological models when quadratic Lagrangians are postulated (Ruzmaikin 1977, Tomita *et al* 1978, Gurovich and Starobinskii 1979, Buchdahl 1978).

We shall now proceed to investigate the existence and properties of Friedman solutions to the general $f(R)$ Lagrangian theories of gravity.

4.1. Existence of Friedman solutions

First, we demonstrate that if the energy–momentum tensor is *trace-free* then *any* homogeneous and isotropic solution of GR is also a solution of an $f(R)$ theory provided $f(0)=0$ and $f'(0) \neq 0$. For, when T^a_a vanishes so does R and hence, by (2.16) and (2.17), we must have

$$f(0)=0 \tag{4.1}$$

and

$$R_{00}f' + \tfrac{1}{2}f = -\rho_0 a^{-4} \tag{4.2}$$

where

$$\rho = \rho_0 a^{-4} = 3p. \tag{4.3}$$

Now, using (2.12) for R_{00} we obtain, if $f'(0) \neq 0$, that

$$\ddot{a} = -\rho_0(3a^3 f'(0))^{-1} \tag{4.4}$$

and this is identical in form to the differential equation for $a(t)$ in GR. Therefore, the cosmological solutions to the $f(R)$ radiation models differ from those of GR only in the definition of the numerical constant. The Friedman radiation solution is an exact solution of any $f(R)$ Lagrangian theory in which $f(0)=0$ and $f'(0) \neq 0$.

4.2. Stability

Now we investigate the conditions necessary for the Friedman radiation solution to be a *stable* solution of an $f(R)$ theory. Equivalently, we might ask when any open set of initial data about the GR Friedman solution will also contain models that evolve towards the GR solution at late times. For simplicity, in this section we shall confine our attention to the flat ($\sigma = 0$) models. To examine the behaviour of perturbations to the Friedman solution we substitute into (2.26)–(2.27) the unperturbed Friedman radiation solution

$$a_0(t) = t^{1/2} \tag{4.5}$$

and derive the perturbation equation from (2.27)–(2.28)

$$\dddot{\varepsilon} + \frac{5}{2t}\ddot{\varepsilon} - \dot{\varepsilon}\left(\frac{f_0'}{3f_0''} + \frac{1}{t^2}\right) - \frac{4\rho_0 \varepsilon}{9tf_0''} = 0. \tag{4.6}$$

If we introduce new variables

$$z = t^{-3/4}(\varepsilon t)^{\cdot} \tag{4.7}$$

$$\tau = (f_0'/3f_0'')^{1/2} t \equiv \lambda t \tag{4.8}$$

then by a redefinition of ρ_0 we may set

$$4\rho_0 = 9f_0''\lambda^2 \tag{4.9}$$

and (4.6) then reduces to the modified Bessel equation

$$\mathrm{d}^2 z/\mathrm{d}\tau^2 + \tau^{-1}\,\mathrm{d}z/\mathrm{d}\tau - (1 + 9/16\tau^2)z = 0 \tag{4.10}$$

which has solutions

$$z = AI_{3/4}(\tau) + BI_{-3/4}(\tau) \qquad A, B \text{ constants} \tag{4.11}$$

where I_p is the modified Bessel function of the first kind, defined in terms of the standard Bessel functions J_p through

$$I_p(x) = \mathrm{i}^{-p} J_p(\mathrm{i}x). \tag{4.12}$$

The asymptotic form of $I_{\pm 3/4}(\tau)$ for large τ gives a solution of the form

$$z \sim \mathrm{e}^{-\tau}(2\pi\tau)^{-1/2}[1 - 5/32\tau + \mathrm{O}(\tau^{-2})] \qquad \text{as } \tau \to \infty. \tag{4.13}$$

Therefore, (4.7) yields

$$\varepsilon \sim t^{-1} \int t^{1/4} \exp(-\lambda t)\,\mathrm{d}t \tag{4.14}$$

and the Friedman solution of GR will be stable if $\lambda^2 > 0$ but unstable if $\lambda^2 \leqslant 0$ (note that λ^2 is only determined by the linear and quadratic pieces of $f(R)$). By way of illustration, consider the quadratic Lagrangian theory when $\Lambda = 0$, so

$$f(R) = R - \tfrac{1}{6}\alpha R^2. \tag{4.15}$$

We have

$$\lambda = (f_0'/3f_0'')^{1/2} = (-\alpha)^{-1/2}. \tag{4.16}$$

Thus, if $\alpha < 0$ we have $\lambda^2 > 0$ and the Friedman GR solution is stable; but if $\alpha > 0$ then λ is imaginary and the Friedman solution is unstable.

These results generalise those of Ruzmaikin and Ruzmaikina (1970) to general $f(R)$ Lagrangians and show that only the quadratic terms influence the stability in this case. Only when $\alpha \leqslant 0$ will the GR Friedman model be approached as $t \to \infty$.

4.3. Existence of singularities

Suppose that, in the general $f(R)$ Lagrangian theory, there exist homogeneous, isotropic cosmological models in which the expansion avoids a singularity and passes through a minimum of $a(t)$. The scale factor will be expressible, for small times, in series form as

$$a(t)=a_0+\tfrac{1}{2}a_1t^2+\tfrac{1}{6}a_2t^3+\cdots+\frac{a_rt^r}{r!}+\cdots \tag{4.17}$$

where the a_i are all constant and $a_0\neq 0$; that is, the model 'bounces' at a minimum radius a_0 and is singularity-free. To determine when such solutions are possible we substitute (4.17) in (2.18) and take the limit as $t\to 0$; we obtain the following existence condition for a solution of the form (4.17) as $t\to 0$

$$0=(1/f_0'')(6f_0'a_0a_1+a_0^2f_0+6T_{00}) \tag{4.18}$$

and so, if $f_0''\neq 0$ and the energy density T_{00} is positive definite, we must have

$$f_0a_0+6a_1f_0'\leqslant 0 \tag{4.19}$$

where $f_0=f(R_0)$ with $R\to R_0$ as $t\to 0$. Using (2.13), we have

$$R_0=-6a_0^{-2}(a_0a_1+\sigma). \tag{4.20}$$

Conditions (4.19) and (4.20) are the necessary conditions for a 'bounce' solution of the form (4.17). As an illustration, again consider the quadratic case (4.15); the bounce conditions (4.19)–(4.20) then reduce to the inequality

$$6\alpha\sigma^2-a_0^2\sigma-6\alpha a_1^2a_0^2<0. \tag{4.21}$$

In particular, *when $\sigma=0$ we have the simple condition $\alpha>0$ for the avoidance of a singularity.* We note that this is linked to the long-time stability of the solution established in (4.16). Any model with $\alpha>0$ avoids a singularity as $t\to 0$, but does not approach the Friedman GR model as $t\to\infty$; in fact, it suffers a future curvature singularity since $a\sim\exp(t^2)$. However, this conclusion is altered when $\sigma\neq 0$; then bounce solutions are possible for both $\sigma=+1$ and $\sigma=-1$ universes. The singularity can be *avoided* if the following conditions hold:

$\sigma=+1$

$$\alpha>0 \text{ and } a_1 \text{ arbitrary} \qquad \text{or } \alpha<-1/6a_1^2 \text{ and } a_1\neq 0 \tag{4.22}$$

$\sigma=-1$

$$\alpha>1/6a_1^2 \text{ and } a_1\neq 0 \qquad \alpha<0 \text{ and } a_1 \text{ arbitrary} \tag{4.23}$$

$\sigma=0$

$$\alpha>0. \tag{4.24}$$

Analogous conditions can be calculated explicitly for any $f(R)$ theory from equations (4.19) and (4.20).

4.4. Horizons

Fischetti *et al* (1979) and Frenkel and Brecher (1982) have looked for horizonless cosmological models in the limit $t\to 0$ when the Lagrangian is quadratic. We can perform similar investigations for the general $f(R)$ theory by seeking solutions to (2.18) of the form

$$a(t) = a_1 t + a_2 t^2 + \cdots + a_r t^r + \cdots \qquad \text{as } t \to 0. \tag{4.25}$$

If, for simplicity, we take the energy–momentum tensor to be that of a perfect fluid (2.19)–(2.20), then the condition for a solution with asymptotic form (4.25) to exist as $t \to 0$ is

$$0 = 2\sigma a_1^2 + 2a_1^4 - \rho_0 a^{1-3\gamma}/6f''(R_0). \tag{4.26}$$

In general, this requires the strong restriction

$$\lim_{t\to 0} \frac{a^{1-3\gamma}}{f''(R_0)} = \text{constant} \tag{4.27}$$

with

$$R_0 = -6a_1^{-2}t^{-2}(a_1^2 + \sigma). \tag{4.28}$$

Consider again the quadratic Lagrangian theory (4.15) where $f_0'' = -\frac{1}{3}\alpha$ is constant. *The condition* (4.27) *can only be satisfied in the quadratic theory if* $\gamma = \frac{1}{3}$; *that is, if the equation of state is that of black-body radiation.* In this case we also require that

$$0 = 4\sigma a_1^2 + 4a_1^4 + \rho_0 \alpha^{-1}. \tag{4.29}$$

Therefore, in the flat and closed models we need the additional condition $\alpha < 0$, to remove horizons as $t \to 0$, and in the open models we require $\alpha > \rho_0 > 0$.

4.5. *Asymptotic behaviour*

At late times the evolution of homogeneous, isotropic cosmological models containing perfect fluids and with $\Lambda = 0$ in the quadratic Lagrangian theory is given by the asymptotic form, as $x \to \infty$, of equation (2.23); that is

$$\mathrm{d}^2 b/\mathrm{d}x^2 + \alpha^{-1} b^{-1/3} x^{-2/3} = 0. \tag{4.30}$$

If we put

$$b = x g(x) \qquad x = \mathrm{e}^z \tag{4.31}$$

then (4.30) becomes

$$\mathrm{d}^2 g/\mathrm{d}z^2 + \mathrm{d}g/\mathrm{d}z + \alpha^{-1} g^{-1/3} = 0. \tag{4.32}$$

The asymptotic form, when $\alpha < 0$, is

$$b(x) \sim (-4/3\alpha)^{3/4} x (\ln x)^{3/4} \qquad x \to \infty \tag{4.33}$$

which corresponds to a scale factor $a(t)$ increasing as

$$a(t) \sim \exp[(t - t_0)^2/4|\alpha|^{2/3}]. \tag{4.34}$$

This solution contains a curvature singularity in the infinite future.

5. Static cosmological models

In this section we shall examine the existence and stability of static solutions to equation (2.18). This means we take the scale factor $a(t)$ to be a constant, a_0, say.

The solutions that then arise generalise the $\sigma = +1$ Einstein static universe and the $\sigma = -1$ Einstein universe of GR.

Equations (2.16) and (2.17) will possess a static solution with scale factor a_0 if

$$R_0 f'_0 - 2f_0 = -(3\gamma - 1)\rho_0 a_0^{-3(1+\gamma)} \tag{5.1}$$

and

$$\tfrac{1}{2} f_0 = -\rho_0 a_0^{-3(1+\gamma)} \tag{5.2}$$

where R_0 is equal to $-6a_0^{-2}\sigma$. By combining (5.1) and (5.2) we find the condition for the existence of this solution is that

$$2R_0 f'_0 = 3(\gamma + 1) f_0. \tag{5.3}$$

Equation (5.2) merely gives a definition of ρ_0 in terms of a_0.

For GR with a non-zero cosmological constant, (2.3), equation (5.3) becomes, for $\gamma \neq -\frac{1}{3}$,

$$R_0 = -6\sigma/a_0^2 = 6(\gamma + 1)\Lambda/(3\gamma + 1). \tag{5.4}$$

When $\gamma = -\frac{1}{3}$ there is only a solution if $\Lambda = 0$, and in that case R_0 is arbitrary. When $\gamma = -1$ or $\Lambda = 0$ and $\gamma \neq -\frac{1}{3}$ we must have $\sigma = 0$ which just gives Minkowski space; otherwise we have

$$a_0^2 = -(3\gamma + 1)\sigma/(\gamma + 1)\Lambda \tag{5.5}$$

yielding a unique closed solution when $(3\gamma + 1)/(\gamma + 1)\Lambda < 0$ and a unique open solution when $(3\gamma + 1)/(\gamma + 1)\Lambda > 0$.

For a quadratic theory (2.4) with $\Lambda = 0$ there are non-trivial solutions determined by the equation (Frenkel and Brecher 1982)

$$R_0 = -6\sigma/a_0^2 = 6(3\gamma + 1)/\alpha(3\gamma - 1) \qquad \gamma \neq \tfrac{1}{3}. \tag{5.6}$$

When $\gamma = \frac{1}{3}$ there is only the trivial solution with $R_0 = \rho_0 = 0$. Equation (5.6) yields a unique, open, closed or flat space–time depending on the sign of α.

We turn now to the stability of these solutions: inserting $a = a_0(1 + \varepsilon)$ into (2.18) where a_0 satisfies (5.1) and (5.2), we find that both the constant and linear order terms vanish identically. The behaviour of the lowest non-vanishing order mode is governed by the quadratic term, so

$$f''_0 \dot{\varepsilon}\dddot{\varepsilon} - \tfrac{1}{2} f''_0 \ddot{\varepsilon}^2 - 2\sigma f''_0 \dot{\varepsilon}^2 - \tfrac{1}{6} f'_0 \dot{\varepsilon}^2 + \tfrac{1}{24}[R_0^2 f''_0 - (\gamma + 1)(3\gamma + 1) f_0]\varepsilon^2 = 0. \tag{5.7}$$

If we seek solutions of the form $\varepsilon \propto \exp(\lambda t)$ then we obtain a quadratic characteristic equation for λ^2. There will exist exponentially growing modes unless both roots of this equation are *negative*, in which case the solutions oscillate but are undamped. If f''_0 is non-zero, both roots will be negative if and only if

$$R_0^2 \geq (1 + \gamma)(1 + 3\gamma) f_0 / f''_0 \tag{5.8}$$

and

$$0 \geq f'_0 / f''_0 + 12\sigma \tag{5.9}$$

while if $f''_0 = 0$ the condition for instability is just

$$0 \geq (1 + \gamma)(1 + 3\gamma) f'_0. \tag{5.10}$$

Equation (5.3) dictates that for flat space to be a solution of the $f(\boldsymbol{R})$ theory we require $f(0)$ zero so (5.8) will always be satisfied. Hence from (5.9) we conclude that flat, static space is unstable if

$$f_0'/f_0'' > 0 \tag{5.11}$$

and will allow undamped oscillations if $f_0'/f_0'' \leq 0$.

These results require careful interpretation: we notice from (2.3) and (5.10) that flat space is unstable even within GR, whereas in the quadratic theory, (2.4), with $\Lambda = 0$, flat space will be unstable if $\alpha < 0$. We recall, from (4.24), that the Friedman solution of the quadratic $f(\boldsymbol{R})$ theory avoided a singularity at $t = 0$, but had a pathological future singularity if $\alpha > 0$. These are equivalent to the condition that flat space be *stable*. By perturbing a static solution to (2.16) and (2.17) we are equating instability with the tendency of a static space to begin expanding or contracting. Hence the GR static universe is clearly expected to be unstable. However, when $\alpha > 0$ static universes will not be unstable in the quadratic Lagrangian theory. This situation requires pathological behaviour of the long-range gravitational field—in effect it needs to increase at large distances—and is associated with the future curvature singularities predicted in the $\alpha > 0$ Friedman solutions. Therefore, we should require of a physically realistic gravity theory derived from an $f(\boldsymbol{R})$ Lagrangian that it satisfy the flat space *instability* criteria (5.8)–(5.11). If these conditions are not satisfied then the theory will possess unphysical long-range behaviour.

A specific example is provided by the Schwarzschild solution stability. Suppose we seek a static, spherically symmetric, vacuum solution to the general field equations (2.9). In the quadratic theory, (2.4), with $\Lambda = 0$ the trace of (2.9) yields a single differential equation for $R(r)$,

$$\nabla^2 R = R\alpha^{-1}. \tag{5.12}$$

When $\alpha < 0$ the solutions are bounded and decay as $r \to \infty$;

$$R(r) = r^{-1}(A\cos(r/\sqrt{-\alpha}) + B\sin(r/\sqrt{-\alpha})) \qquad A, B \text{ constant.} \tag{5.13}$$

However, when $\alpha > 0$ they are singular as $r \to \infty$. The Schwarzschild solution of GR corresponds to choosing the particular solution $R = 0$ in both (5.12) and (5.13); it is *unstable* when $\alpha > 0$.

It appears that when $\alpha > 0$ the action variation (2.2) leads to a stationary action ($\delta S/\delta g_{ab} = 0$) that is *non-minimal* (Ruzmaikin and Ruzmaikina 1970):

$$\delta^2 S < 0. \tag{5.14}$$

When $\alpha < 0$ in the quadratic theory, the stationary action is also minimal. We conjecture that a necessary condition for the solutions of an $f(\boldsymbol{R})$ Lagrangian gravity theory to be physically realistic is that the action functional variation be both stationary and minimal.

6. Conclusions

We have analysed the behaviour of homogeneous and isotropic solutions to the gravity theory that arises from the variation of an arbitrary analytic function of the scalar curvature, $f(R)$. Such a theory generalises Einstein's general relativity (which arises when f is a linear function of R) and provides insight into the possible consequences of adding quantum corrections to the usual Einstein equations in the regime where $R \to \infty$. We have ascertained the extent to which homogeneous and isotropic cosmological solutions of the $f(R)$ theory resemble those of general relativity at early and late cosmic epochs.

The conditions were found under which the de Sitter, Friedman and Einstein static cosmological models were also solutions to the $f(R)$ theory; and, in addition, when the solutions were stable, that is, when they approached the solutions of Einstein's theory at late times as astronomical observations dictate they must. We also found necessary and sufficient conditions on f for the existence of initial big bang singularities and particle horizons in the cosmological solutions to the $f(R)$ theory.

Our study of the asymptotic behaviour of the cosmological solutions as $t \to \infty$ revealed that in general those models which avoid an initial singularity have pathological late time behaviour with $R \to \infty$ as $t \to \infty$. These pathological theories also exhibit an instability for Minkowski space and Schwarzschild space. All these pathologies can be traced back to the fact that the field equations have arisen from an action functional whose variation is stationary but non-minimal.

Finally, it is instructive to point out the similarity of some of our findings to the properties of singular perturbation problems of the sort that are common in hydrodynamics (O'Malley 1974). The field equations (2.16)–(2.18) that arise when higher-order curvature terms are added to Einstein's Lagrangian have the typical form exhibited by equations that possess singular perturbative properties; that is, they contain higher-order derivatives multiplied by some small parameter. In some cases we have found that the addition of higher-order curvature terms to the Einstein theory produces radically different behaviour at late times in cosmological solutions, even though dimensional analysis would suggest they are negligible in that limit. A simple example is provided by the equation

$$-\varepsilon\ddot{y} + \dot{y} + y = 0 \qquad 0 \leq \varepsilon \ll 1. \tag{6.1}$$

When ε is zero the equation has exponentially decaying solutions

$$y = e^{-t} \tag{6.2}$$

but when $\varepsilon \neq 0$ the general solution grows exponentially with time with an exponent that varies inversely with the small parameter

$$y \sim e^{+t/\varepsilon}. \tag{6.3}$$

Dimensional analysis would have suggested that the $\varepsilon\ddot{y}$ term is negligible with respect to y and $\dot{y}$ at large times. This type of behaviour can be seen explicitly in the solution of (4.30) given by (4.34): the approach to Einstein's theory through the limit $\alpha \to 0$ is singular. In some cases we have found such singular behaviour to be associated with unphysical aspects of Lagrangian theory of gravity, but that is by no means necessarily the case. The presence of singular perturbative behaviour in simple hydrodynamic problems shows that it is not just a manifestation of bad modelling or unphysical boundary conditions. Perhaps the possibility of such a situation offers a ray of hope to quantum gravity theorists seeking some present-day consequence of quantum modifications to the Einstein equations at the Planck epoch.

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